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Discount Rates for Public Enterprises in the Presence of Alternative Financial Constraints

By

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1. Introduction

In this article we develop formulae for optimal discount rates to be employed by a public enterprise facing a financial constraint. Optimality here refers to a situation where the government chooses values for the instruments at its disposal in order to obtain a second-best optimum. The focus of our study is on the changes we observe in these discount rate formulae as we alter the nature of the financial constraint. Hagen (1982) studies the same phenomenon; however, he performs a different set of comparisons[1]. Pestieau (1979) is also concerned with the consequences for discount rate formulae of altering the economic environment, but in only one of his options is the public enterprise subject to a budget constraint.

The alternative models we consider are, to use Guesnerie's (1980) terminology, "in the Boiteux tradition". In the spirit of Boiteux (1956) we envisage our public firm facing certain restric-

* We are indebted to the conference participants, particularly John Kay and Kåre Hagen, and an anonymous referee for their comments.

[1] Hagen's manuscript appeared while we were preparing the final draft of our article. While there is some overlap with our work, his discount rate formulae are obtained by employing substantially different manipulations to the basic first-order conditions for optimality. We have limited our remarks on his article to pointing out some of the major respects in which our analyses differ. Space limitations prevent a systematic comparison of our results; such a comparison, we believe, would lead to a deeper understanding of the issues we address.

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tions on the size of the losses that it can legally incur. This constraint may be over the lifetime of the enterprise or may take the form of a series of constraints, one for each period. Furthermore, the public enterprise may or may not have access to government tax revenues to keep its losses within the bounds allowed by law. These various possibilities are the source of our alternative financial constraints[2].

Optimal pricing rules for such firms have been studied by Bernard (1977), Boiteux (1956), Drèze and Marchand (1976), Guesnerie (1980), and Hagen (1979) among others. However, in these contributions little attention is devoted to an analysis of the discount rates implicit in the various pricing formulae. We believe that the explicit development of these discount rate formuale is a worthwhile endeavor. Because of the comprehensiveness of Guesnerie's study of optimal pricing rules, we allocate little space to pricing rules, as opposed to discount rate formulae. The range of possible environments considered here is, however, modest in comparison with Guesnerie's study.

In modelling our constraints and interpreting our results we have been strongly influenced by Hagen's articles. Hagen is one of the few contributors to the Boiteux literature to explicitly study discount rates. In Hagen (1978) a model is considered which permits the public enterprise to finance some of its losses by means of commodity taxes; his model is discussed in section 5.2. The variant of this model presented in section 5.1 is also analyzed in Hagen (1982)[3].

We find, as have many others, that our formulae for the public enterprise's shadow discount rate can be expressed as a weighted sum of various discount rates found in the economy. One feature of our formulae is the use of a discount rate that is implicitly defined by the multipliers associated with the resource constraints on the economy. These multipliers are interpreted to be social values of commodities in Guesnerie (1980) and Tirole (1981) but, through a series of manipulations, have generally been eliminated from the existing formulae in the discount rate literature.

The fact that it is natural to express public discount rate formulae in terms of a weighted sum of other discount rates in the economy does not seem to be well-explained in the literature. That

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[2] The literature on the social discount rate is vast, dealing with a number of conceptually different problems. Tirole's (1981) survey sorts out the main issues.

[3] The public enterprises studied by Hagen produce public as well as private goods. We restrict attention to private goods production.

the formulae should take this form follows simply from the relationships that must hold between gradients of objectives and constraints at any constrained optima. For example, suppose we have a single objective with two constraints as shown in Fig. 1, where the shaded area depicts the feasible set. The gradients of the constraints are labelled a and b while the gradient of the objective function is denoted by c. Optimality requires that non-negative weights λ_1 and λ_2 can be found such that $c=\lambda_1 a+\lambda_2 b$; i. e., the gradient of the objective function is a weighted sum of the gradients of the constraints[4]. Thus, as a consequence of optimality, weight-sum formulae will appear.

In the context of the problem we are considering, the public firm will be facing two constraints: (1) a production constraint whose gradient vector is a set of shadow prices and (2) a financial constraint which, in the relevant dimensions, has either the producer or consumer price vector as a gradient. In effect, our optimization problems will be implicitly asking the public enterprise to maximize profits using social values as prices. Thus these social values can be expressed as a weighted sum of the public enterprise's shadow prices and either the consumer or producer price vector. Since discount rates are obtained by taking ratios of prices, this simple weighted-sum formula will be preserved when the pricing rules are reinterpreted in terms of discount rates.

In section 2, we present the model to be employed throughout the remainder of this article.

In section 3, we develop discount rate formulae for the basic Boiteux (1956) firm. There is no commodity taxation and the public enterprise has a lower limit on the present-discounted value of its profits, which would typically be non-positive. If this lower limit is zero, the public enterprise has a break-even constraint.

In section 4, we alter the financial constraint so that it applies period by period, still without any commodity taxes. We observe that our discount rate formulae are altered by the inclusion of a term which takes into account the redistribution of profits across time that occurs when prices change.

In section 5, we return to the case of the public enterprise facing a single constraint but allow the firm to keep within its budget by allowing it access to commodity tax revenue. Permitting tax rates to vary would increase the set of policy variables compared to the models considered in the previous sections. As we are inter-

[4] See Guesnerie (1979) for a formal statement of the relevant duality theorems.

ested in the consequences of varying the financial constraint and not in varying the set of policy instruments, we assume the tax rates are fixed. Hagen (1978, 1982) presents discount rate formulae when taxes are set optimally. Commodity taxes could be levied on either the transactions of the household sector or on the transactions of the private firms; both possibilities are considered. In the first case our formulae for the public firm's discount rate will contain terms utilizing the producer interest rate while in the second case the consumer interest rate will appear. We shall show that in fact the optimal solution to our problem does not depend upon which method of tax collection is used. In effect, we have obtained two different methods of decomposing a discount rate into constituent parts. We believe that this observation, and the lessons to be learned from it, are of considerable importance in interpreting discount rate formulae.

Section 7 presents some concluding remarks.

2. The Model

The essential features of our problem are captured in a three-good model; the commodities are indexed by $i=1, 2, 3$. For interpretive purposes we shall suppose that there are two time periods. Goods one and three are physically the same commodity, available in the first and second periods respectively. Good two is first-period labour or leisure. Firms will use inputs of labour and commodity one in period one to produce commodity two for second-period consumption. Households will (in the aggregate) supply firms with labour and commodity one in period one, purchasing commodity two for delivery in period two.

There are H households who are assumed to have continuous, strictly quasi-concave, locally non-satiated preferences which shall be represented by differentiable indirect functions $U^h(q, R^h)$, where $q=(q_1, q_2, q_3)$ is a vector of present-value consumer prices and R^h is household h's lump-sum income. Households also have endowments of goods, but they will not be considered explicitly as we shall only be concerned with the net (of endowment) demand functions,

$$x_i^h = x_i^h(q, R^h) \qquad h=1,\ldots, H; \quad i=1, 2, 3. \tag{2.1}$$

These net demand functions are assumed to be differentiable. A negative value for x_i^h represents household h's supply of commodity i. By the local non-satiation assumption, each household will exhaust its lump-sum income. Aggregate household net demand for good i will be denoted by $x_i = \sum_h x_i^h$.

All firms in the private sector are assumed to behave competitively and face the same prices so there is no loss in generality in only considering an aggregate firm. The aggregate supply functions are

$$y_i = y_i(p) \qquad i = 1, 2, 3, \tag{2.2}$$

where $p = (p_1, p_2, p_3)$ is a vector of present-value producer prices. If y_i is negative, the private sector is using good i as an input.

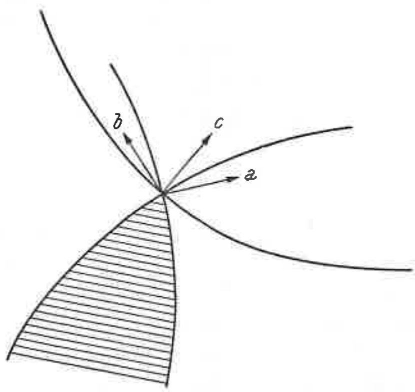

Fig. 1

For simplicity, we shall only consider a single public firm. It chooses a production plan $z = (z_1, z_2, z_3)$ subject to a technology constraint,

$$F(z_1, z_2, z_3) \leq 0. \tag{2.3}$$

In keeping with the Boiteux (1956) model, this technology is assumed to exhibit increasing returns over part of its domain. For example, the public firm could have a standard convex technology shifted from the origin by the presence of set-up costs. F is assumed to be differentiable in the neighbourhood of any optima we consider.

In addition to this technology constraint, the public firm will be assumed to face a financial constraint, e. g. a non-negative profitability condition. A variety of such constraints will be considered in subsequent sections.

Feasibility will require that aggregate household net consumption of each commodity does not exceed total production,

$$x_i \leq y_i + z_i \qquad i=1, 2, 3. \tag{2.4}$$

We shall be interested in properties of discount rates at Pareto-optimal allocations. For this reason, we take as our objective function

$$\sum_h \mu^h U^h (q, R^h). \tag{2.5}$$

Different Pareto-optimal allocations are obtained by varying the weights, μ^h.

First-best optima are found by maximizing (2.5) subject to the constraints (2.1), (2.2), (2.3) and (2.4)[5]. However, if in addition to these constraints, the public firm also faces a financial constraint, the resulting allocation will typically be a second-best optimum.

In subsequent sections it will be convenient to substitute (2.1) and (2.2) directly into (2.4), obtaining

$$\sum_h x_i^h (q, R^h) \leq y_i (p) + z_i \qquad i=1, 2, 3. \tag{2.6}$$

This is simply the condition that demand does not exceed supply.

The government's budget constraint need not be considered explicitly as it will automatically be satisfied as a consequence of Walras' Law. Since there are two sets of prices, the government budget constraint can be written in two different forms. Multiplying each of the inequalities in (2.4) by the corresponding consumer price and adding yields

$$\sum_i q_i z_i + \sum_i t_i y_i + \sum_i p_i y_i \geq \sum_h R^h \tag{2.7}$$

where

$$t_i = q_i - p_i \qquad i=1, 2, 3, \tag{2.8}$$

is the per-unit tax (or subsidy) on commodity i. In (2.7) use has been made of the fact that total household income will equal the total value of households' net demands. Alternatively, using producer prices we obtain

$$\sum_i p_i z_i + \sum_i t_i y_i x_i + \sum_i p_i \geq \sum_h R^h. \tag{2.9}$$

In each case, the government has three sources of revenue to finance the lump-sum income transfers. First, there are the profits of the public firm. Second, there is the revenue raised from com-

[5] Throughout we shall assume that we have interior optima and thus ignore non-negativity constraints.

modity taxation. Third, private profits are taxed at a 100% rate[6]. In (2.7) the commodity taxes are levied on the sales and purchases of the private firms with public profits valued in consumer prices. In (2.9) commodity taxes are levied on consumer transactions with the public firm's profits valued in producer prices.

As noted previously, in our three-good model it is natural to suppose that x_1 and x_2 will be negative. If these goods are taxed (as opposed to subsidized), both t_1 and t_2 must be negative because of our sign conventions.

Each of the models considered in the following sections uses the basic model outlined above supplemented by a financial constraint for the public firm. In section 3 and 4 we shall suppose that no commodity taxation is possible; consumers and producers will then face common prices which shall be denoted by p. As multipliers on the constraints we shall use γ for (2.3) and ϱ_i, $i=1, 2, 3$ for (2.6). The multipliers for (2.6) are the shadow prices associated with having an additional unit of commodity i available to the community; they are often referred to in the second-best literature as social values.

We have four sets of prices which are of interest: (1) the vector of producer prices p, (2) the vector of consumer prices q, (3) the vector of social values ϱ, and (4) the shadow prices which (locally) support the production decision of the public firm and are given by the gradient vector $\nabla F = (F_1, F_2, F_3)$, where subscripts indicate partial differentiation. As a normalization, we set $p_3 \equiv 1$. From these prices we can determine four sets of discount rates[7]:

$$1+r_i \equiv p_i/p_3 \qquad i=1, 2, 3 \tag{2.10}$$

$$1+s_i \equiv q_i/q_3 \qquad i=1, 2, 3 \tag{2.11}$$

$$1+\alpha_i \equiv \varrho_i/\varrho_3 \qquad i=1, 2, 3 \tag{2.12}$$

and

$$1+\delta_1 \equiv F_i/F_3 \qquad i=1, 2, 3 \tag{2.13}$$

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[6] This assumption about profit taxation involves no loss of generality in the presence of unrestricted lump-sum redistributions. In this article we suppose that such redistributions are possible, i. e., we focus on efriciency questions. The modifications required in optimal pricing formulae when this assumption is dropped are wellknown; similar changes would apply to our formulae.

[7] We shall, rather loosely, refer to both, for example, r_i and $1+r_i$ as discount rates. By taking the ratio of the price of good one and the price of good two we obtain yet another set of discount rates but they are of no direct interest here.

Of course, in each case when $i=3$ these expressions are equal to one. Of particular interest are the own-rates-of-return which are obtained by considering $i=1$.

3. Public Firm Subject to a Profit Constraint

In this section we shall consider the standard Boiteux (1956) model of the public firm. No commodity taxation is possible so $t=0$. The public enterprise is required to meet a profitability constraint,

$$\sum_i p_i z_i \geq b. \tag{3.1}$$

When $b=0$, (3.1) says that the present discounted value of the public firm's profits must be non-negative. With a region of increasing returns, in a first-best optimum it will generally be desirable for a public firm to run a loss, with the deficit recovered by lump-sum taxes. Thus, such a financial constraint introduces a second-best element into our problem.

In (3.1) we do not require b to be zero. If it is negative, the firm's losses must not exceed $|b|$. This lower limit on profits is expressed in monetary terms, which is only appropriate because we have normalized prices. This normalization avoids the possibility of meeting the financial constraint by scaling all prices proportionally.

Without the possibility fo commodity taxation, the overall government budget constraint simplifies to

$$\sum_i p_i z_i + \sum_i p_i y_i \geq \sum_h R^h. \tag{3.2}$$

Total lump-sum income must not exceed the total profits obtained from public production and private production.

In the first period the public enterprise will incur a debt equal in magnitude to the amount $p_1 z_1 + p_2 z_2$ while private firms have debts equal in magnitude to $p_1 y_1 + p_2 y_2$. These funds are borrowed from the household scetor in the capital market implicit in this model. In the second period the loans are repaid. If $b<0$, general government revenues are used to help pay back the loans while if $b>0$, the public firm contributes this amount to general government revenues. One can easily demonstrate that the government budget constraint (3.2) is equivalent to the requirement that there be equilibrium in the capital market[8].

[8] See Hagen (1978, 1982) for a proof as well as a more detailed discussion of the capital market.

The government wishes to maximize (2.5) subject to the constraints (2.3), (2.6) and (3.1). Its instruments are the variable prices p_1 and p_2, the lump-sum incomes R^h, and the production decision of the public enterprise z. Here, and in the remaining sections, β will be the multiplier on the financial constraint. We shall also assume that all constraints are binding.

The first-order conditions are

$$\sum_h \mu^h \frac{\partial U^h}{\partial p_k} - \sum_i \varrho_i \left(\sum_h \frac{\partial x_i^h}{\partial p_k} - \frac{\partial y_i}{\partial p_k} \right) + \beta z_k = 0 \qquad k=1,2, \tag{3.3}$$

$$\mu^h \frac{\partial U^h}{\partial R^h} - \sum_i \varrho_i \frac{\partial x_i^h}{\partial R^h} = 0 \qquad h=1,\ldots,H, \tag{3.4}$$

and

$$\varrho_k + \beta p_k - \gamma F_k = 0 \qquad k=1,2,3. \tag{3.5}$$

These first-order conditions yield necessary conditions for the optimality of public production decisions as well as the optimality of the prices p and the lump-sum incomes R^h. These conditions are merely necessary, and not sufficient, for optimality since our problem is non-convex[9].

These conditions can be simplified by introducing

$$e_i = x_i - y_i \qquad i=1,2,3 \tag{3.6}$$

and

$$\frac{\partial e_i}{\partial p_k} \equiv \sum_h \left. \frac{\partial x_i^h}{\partial p_k} \right|_{U^h} - \frac{\partial y_i}{\partial p_k} \qquad i=1,2,3;\ k=1,2, \tag{3.7}$$

where $\left. \frac{\partial x_i^h}{\partial p_k} \right|_{U^h}$ is a compensated demand derivative. When the feasibility constraint is binding, e_i, the private net demand for good i, is equal to the public supply of good i, z_i. Using (3.7) and Roy's identity, (3.3) and (3.4) imply

$$\sum_i \varrho_i \frac{\partial e_i}{\partial p_k} = \beta z_k \qquad k=1,2, \tag{3.8}$$

which is a well-known optimality formula for a Boiteux firm[10]. Eq. (3.8) represents a balancing of the benefits and costs of chang-

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[9] Harris (1981) provides an extended discussion of the issues raised by restricting attention to necessary conditions for optimality in the presence of non-convexities.

[10] For example, see Guesnerie (1980, Eq. (39), p. 63). Guesnerie also considers a number of ways in which (3.8) can be manipulated to yield interesting interpretations.

conditions in terms of compensated demands let us suppose that lump-sum incomes are adjusted to keep utilities constant. To obtain these extra resources, the price of commodity one will be bid up so r_1 increases. The denominator on the right-hand side of (3.18), which is negative, is equal in magnitude to the extra input of good one made available to the public firm. For this endeavor to be socially worthwhile each additional unit of this resource must yield a gross return of $1+\delta_1$. Thus the numerator in (3.18) provides a decomposition of the gross rate of return required for an additional unit of good one employed by the public enterprise.

The first two terms in this decomposition correspond to what Hagen (1979, p. 484) refers to as the reallocation effect. Together these terms measure the change in the value of the compensated net private sector demand in the markets for goods two and three induced by the interest rate change in the market for good one. The first term captures the social benefit or cost a compensated change in the interest rate will have on the labour market, while the second term captures the effect on the market for the second-period commodity. Since equilibrium is preserved, these two terms are equal to the change in the social value of public production. This result can also be seen by observing that with our normalizations, $1+\delta_2$ is the social value of the marginal product of labour in public production and $1+\delta_3 \equiv 1$ is the social value of additional second-period output. Thus, the first two terms in (3.18) capture the effects that compensated adjustments in the interest rate r_1 will have on behaviour in the other two markets.

The third term reflects the fact that a change in r_1 and, hence, in p_1 will result in a revaluation of the firm's production plan. To a first-order approximation, the public firm's profits change by the amount z_1. With z_1 negative this increases the firm's losses; the social cost of this drop in profits is obtained by using the multiplier $\bar{\beta}$, which measures the cost of violating the financial constraint. Hence, this third term tends to increase the rate of return necessary to make an increase in activity worthwhile.

An alternative decomposition of δ_1 is possible using (3.17),

$$\delta_1 = r_1 + (r_2 - \delta_2) \frac{\dfrac{\partial e_2}{\partial r_1}}{\dfrac{\partial e_1}{\partial r_1}} + \frac{\bar{\beta} z_1}{\dfrac{\partial e_1}{\partial r_1}}. \tag{3.19}$$

In (3.19) we view the public enterprise's shadow discount rate as the private discount rate with suitable adjustments. As in (3.18), the last term reflects the revaluation effect a compensated change in the interest rate has on profits. The second term adjusts for distortions in the labour market; the labour supply response is valued by the difference between the private and shadow public rates of return.

With three markets, knowledge of behaviour in two markets completely determines behaviour in the third. In (3.18) the decomposition considered the behavioural consequences of a compensated interest rate adjustment on the labour market and the market for the second-period commodity. The need for explicitly considering the latter is eliminated in (3.19) by the first term which accounts for changes in the market for commodity one.

If the compensated private net demand for labour is completely insensitive to the interest rate r_1, the second term in (3.19) drops out. In this case, the only source of divergence between the public and private discount rates is provided by the adjustment for the revaluation of profits caused by a change in the interest rate. With the financial constraint binding, $\bar{\beta} > 0$ which implies that the third term is positive since both z_1 and $\partial e_1/\partial r_1$ are negative. Consequently, in these circumstances the shadow discount rate δ_1 is unambiguously larger than the private discount rate r_1. Using our previous remarks concerning (3.11) and (3.14) we may also conclude that δ_1 is unambiguously smaller than the social value discount rate α_1. Having δ_1 larger than r_1 merely reflects the fact that the binding financial constraint results in the social opportunity cost of resources being higher in the public sector than in the private, a point which has been noted in the optimal pricing literature[13].

With $\partial e_2/\partial r_1 = 0$, four pieces of information are needed to calculate the shadow discount rate δ_1: r_1, z_1, $\partial e_1/\partial r_1$, and $\bar{\beta}$. Both r_1 and z_1 are easily observed and $\delta e_1/\partial r_1$ is, in principle, possible to obtain. However, $\bar{\beta}$ is a multiplier which is not observable, just as α_1 is an unobservable social value discount rate. It would seem that α_1 is not a practical upper bound for δ_1 nor is (3.19) a formula which can be computed using observable data. However, we do know that $0 \leq \bar{\beta} \leq 1$ so that it is possible to obtain an observable upper bound for δ_1 by setting $\bar{\beta} = 1$. If the shadow discount rate in actual use did not fall in the range $\left(r_1, r_1 + \dfrac{z_1}{\dfrac{\partial e_1}{\partial r_1}} \right)$, it has been inappropriately chosen.

[13] For example, see Drèze and Marchand (1976, p. 67) or Hagen (1979, p. 481). Having $\partial e_2/\partial r_1 = 0$ is easily shown to result in inverse-elasticity pricing rules.

However, if $\frac{\partial e_2}{\partial r_1} \neq 0$ it is much more difficult to obtain observable bounds for δ_1. One problem is the appearance of the other shadow discount rate δ_2 on the right-hand-side of (3.19). Of course we could have obtained an expression similar to (3.19) for δ_2 by setting $k=2$ in (3.17), i. e., investigate the consequences of obtaining resources for the public sector by bidding up wage rates, which would then allow us to solve for δ_1 and δ_2 simultaneously in terms of observables and the multiplier $\bar{\beta}$. However, once we expand the number of commodities past three this procedure becomes extremely difficult in practice, suggesting that formulae such as (3.18) or (3.19) are of most practical use when cross-elasticities are small.

4. Public Firm Subject to a Profit Constraint Each Period

The financial constraint considered in the previous section only limited the lifetime profitability of the public enterprise. In practice, separate financial constraints are often imposed for each time period. In this section we modify the problem considered in section 3 by replacing the single financial constraint by the pair of constraints,

$$p_1 z_1 + p_2 z_2 \geq b_1 \tag{4.1}$$

and

$$z_3 \geq b_2, \tag{4.2}$$

with multipliers β_1 and β_2 respectively. Thédié (1977) and Tirole (1981) have considered period-by-period constraints on the value of public enterprise inputs. For (4.1) and (4.2) to be binding, and to make the problem non-trivial, b_1 would be negative since the public firm only employs inputs in period one while b_2 would be positive. If over time the firm was required to break even, we would have $b_1 + b_2 = 0$.

Necessary conditions for second-best optimality are now found by maximizing (2.5) subject to (2.3), (2.6), (4.1) and (4.2). The available instruments are the same as in section 3. The resulting first-order conditions are (3.4),

$$\sum_h \mu^h \frac{\partial U^h}{\partial p_k} - \sum_i \varrho_i \left(\sum_h \frac{\partial x_i^h}{\partial p_k} - \frac{\partial y_i}{\partial p_k} \right) + \beta_1 z_k = 0, \qquad k=1, 2, \tag{4.3}$$

$$\varrho_k + \beta_1 p_k - \gamma F_k = 0 \qquad k=1, 2 \tag{4.4}$$

and

$$\varrho_3 + \beta_2 - \gamma F_3 = 0. \tag{4.5}$$

These conditions only differ from the ones developed in section 3 in the use of separate multipliers for the two financial constraints. Combining (3.4) and (4.3) we obtain

$$\sum_i \varrho_i \frac{\partial e_i}{\partial p_k} = \beta_1 z_k \qquad k=1, 2. \tag{4.6}$$

Our first expressions for the public firm's shadow discount rates are obtained from (4.4) and (4.5),

$$1+\delta_i = \frac{\beta_1 p_i}{\beta_2+\varrho_3} + \frac{\varrho_i}{\beta_2+\varrho_3} \qquad i=1, 2 \tag{4.7}$$

or

$$1+\delta_i = \left(\frac{\beta_1}{\beta_2+\varrho_3}\right)(1+r_i) + \left(\frac{\varrho_3}{\beta_2+\varrho_3}\right)(1+\alpha_i) \qquad i=1, 2. \tag{4.8}$$

As in the case with a single financial constraint, the shadow discount rate can be expressed as a weighted sum of the market discount rate and the discount rate implicitly defined by the social value of commodities. However, there is no reason to expect β_1 to equal β_2, so the weights will not, in general, sum to one. Consequently, r_i and α_i no longer provide bounds for δ_i.

We may define normalized multipliers

$$\bar{\beta}_i \equiv \frac{\beta_i}{\gamma F_3} \qquad i=1, 2. \tag{4.9}$$

Unfortunately, the procedure used to bound $\bar{\beta}$ in section 3 only permits us to conclude that $0 \leq \bar{\beta}_2 \leq 1$; we are not able to provide an upper bound for $\bar{\beta}_1$.

Social values may be eliminated by proceeding as we did in the previous section. Corresponding to (3.18) is

$$1+\delta_1 = -\frac{(1+\delta_2)\dfrac{\partial e_2}{\partial r_1} + \dfrac{\partial e_3}{\partial r_1} + (\bar{\beta}_1-\bar{\beta}_2)\dfrac{\partial e_3}{\partial r_1} - \bar{\beta}_1 z_1}{\dfrac{\partial e_1}{\partial r_1}} \tag{4.10}$$

Aside from the fact that the last term uses $\bar{\beta}_1$, the only difference between (4.10) and (3.18) is the introduction of the term $(\bar{\beta}_1-\bar{\beta}_2)\,\partial e_3/\partial r_1$ in the former. This expression takes account of the benefits and costs of the redistribution of profits across time periods that results when demands and supplies respond to the change in

The first-order conditions are (3.4),

$$\sum_h \mu^h \frac{\partial U^h}{\partial q_k} - \sum_i \varrho_i \left(\sum_h \frac{\partial x_i^h}{\partial q_k} - \frac{\partial y_i}{\partial p_k}\right) + \beta \left(z_k + \sum_i t_i \frac{\partial y_i}{\partial p_k}\right) = 0, \qquad k=1,2 \tag{5.3}$$

and

$$\varrho_k + \beta q_k - \alpha F_k = 0, \qquad k=1,2,3. \tag{5.4}$$

Consumer prices appear in (5.4) since public enterprise profits are valued in consumer prices.

Using (3.7) and Roy's Identity, (5.3) and (3.4) imply

$$\sum_i \varrho_i \frac{\partial e_i}{\partial q_k} = \beta \left(z_k + \sum_i t_i \frac{\partial y_i}{\partial p_k}\right) \qquad k=1,2. \tag{5.5}$$

The only significant difference between (5.5) and (3.8) is the term $\beta \sum t_i \frac{\partial y_i}{\partial p_k}$. With the introduction of commodity taxes, the social cost of the change in tax revenue resulting from the supply response to a change in prices must be added to the revaluation of profits term. The term in square brackets in (5.5) is simply the derivative of the left-hand side of (5.2), the financial constraint, with respect to p_k [15].

From (5.4) we can develop our first discount rate formulae,

$$\delta_i = \left(\frac{\beta q_3}{\beta q_3 + \varrho_3}\right) s_i + \left(\frac{\varrho_3}{\beta q_3 + \varrho_3}\right) \alpha_i \qquad i=1,2,3. \tag{5.6}$$

Eq. (5.6) expresses the shadow discount rate for the public enterprise as a weighted sum of the discount rate for consumers and the discount rate obtained from the social values of commodities. The appearance of the consumer, rather than the producer, discount rate follows from the form of the financial constraint (5.1); the public firm's profits are valued in consumer prices, implicitly stating that borrowing is undertaken using consumer interest rates. The weights in (5.6) are non-negative and sum to one (since there is only one financial constraint), which implies that the shadow discount rates δ_i are bounded by the consumer discount rates s_i and the social value discount rates α_i.

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[15] Hagen (1982) substitutes the market clearing condition (2.6), assumed to hold with equality, directly into the financial constraint. Because of this, the development of his discount rate formulae differs substantially from our own.

Using the manipulations employed in section 3, from (5.5) and (5.6) we obtain

$$1+\delta_1 = -\frac{(1+\delta_2)\dfrac{\partial e_2}{\partial s_1} + \dfrac{\partial e_3}{\partial s_1} - \bar{\beta}\left(z_1 + \sum\limits_i t_i \dfrac{\partial y_i}{\partial r_1}\right)}{\dfrac{\partial e_1}{\partial s_1}}. \tag{5.7}$$

In (5.7) the public firm increases its command over inputs by increasing the consumer interest rate s_1 which, since taxes are fixed, increases the producer interest rate r_1, as well. Eq. (5.7) can be interpreted in the same fashion as was done for (3.18). The presence of taxes merely adds the term $\bar{\beta} \sum\limits_i t_i \, \partial y_i / \partial r_1$ to the numerator, capturing the effect on the financial constraint discussed in relation (5.5).

Alternatively, the shadow discount rate can be viewed as an adjusted form of the consumer discount rate,

$$\delta_1 = s_1 + (s_2 - \delta_2)\frac{\dfrac{\partial e_2}{\partial s_1}}{\dfrac{\partial e_1}{\partial s_1}} + \frac{\bar{\beta}\left(z_1 + \sum\limits_i t_i \dfrac{\partial y_i}{\partial r_1}\right)}{\dfrac{\partial e_1}{\partial s_1}}. \tag{5.8}$$

If $\partial e_2/\partial s_1$ is zero, the second term in (5.8) drops out. If, in addition, increasing the interest rate r_1 decreases commodity tax revenue, the last term in (5.8) will be positive. When both of these conditions are satisfied, the shadow discount rate δ_1 is unambiguously larger than the consumer interest rate s_1. Furthermore, since $\bar{\beta} \leq 1$ we may, as was done in section 3, obtain an upper bound for δ_1 which can be calculated from observable phenomenon when $\partial e_2/\partial s_1$ is zero.

5.2 Taxes on Consumers' Transactions

We now suppose that commodity taxes are levied on consumers' transactions. The financial constraint facing the public enterprise is

$$\sum_i p_i z_i + \sum_i t_i x_i \geq b \tag{5.9}$$

which becomes

$$\sum_i p_i z_i + \sum_i t_i \sum_h x_i^h (q, R^h) \geq b \tag{5.10}$$

upon the substitution of (2.1) for x_i. For brevity, we omit the intermediate steps and proceed directly to our discount rate formulae[16].

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[16] See Hagen (1978) for a detailed derivation of a similar model.

Our first formula is

$$\delta_i = \left(\frac{\beta}{\beta+\varrho_3}\right) r_i + \left(\frac{\varrho_3}{\beta+\varrho_3}\right) \alpha_i, \qquad i=1, 2, 3, \tag{5.11}$$

where use has been made of our normalization $p_3 \equiv 1$.

The shadow discount rate for the public enterprise is a convex combination of the private firm's discount rate and the social value discount rate. With the financial constraint (5.9), the public firm borrows on the same terms as the private firm, which results in the use of the producer's discount rate in (5.11). Again, we have bounds for δ_i, namely r_i and α_i.

Analogous to (5.7) we obtain

$$1+\delta_1 = -\frac{(1+\delta_2)\frac{\partial e_2}{\partial s_1} + \frac{\partial e_3}{\partial s_1} - \bar{\beta}\left(z_1 + \sum_i t_i \frac{\partial x_i^c}{\partial s_1}\right)}{\frac{\partial e_1}{\partial s_1}}, \tag{5.12}$$

where

$$\frac{\partial x_i^c}{\partial s_1} \equiv \Sigma \frac{\partial x_i^h}{\partial s_1}\bigg|_{U^h}. \tag{5.13}$$

It is inessential whether these derivatives are calculated with respect to s_1 or r_1 since the tax rates are fixed.

The new feature in (5.12) is the term $\bar{\beta}\, \Sigma\, t_i \frac{\partial x_i^c}{\partial s_1}$. Recall that we have developed our formulae for interest rate changes which are income-compensated to hold utilities constant. This new term thus measures the social cost of the change in tax revenue caused by the households' demand response, and corresponds to the effect discussed in relation to (5.5). Taking the derivative of the left-hand side of the financial constraint (5.10) yields the term in square brackets in (5.12).

Viewing the public enterprise shadow discount rate as an adjusted producer discount rate, we have

$$\delta_1 = r_1 + (r_2 - \delta_2)\frac{\frac{\partial e_2}{\partial s_1}}{\frac{\partial e_1}{\partial s_1}} + \bar{\beta}\frac{\left(z_1 + \sum_i t_i \frac{\partial x_i^c}{\partial s_1}\right)}{\frac{\partial e_1}{\partial s_1}} \tag{5.14}$$

Our remarks in the previous subsection concerning bounds for δ_1 have straightforward analogues here.

5.3 A Comparison of the Tax Institutions

Because of the appearance of consumer discount rates in the formulae of subsection 5.1 and producer discount rates in the formulae of subsection 5.2, one might at first believe that this would lead to quite different values for the shadow discount rates, depending upon which of the two methods of tax collection is adopted. In fact this is not the case. Assuming that all constraints are binding, the optimal solutions for the real variables do not vary with the tax institutions nor do the various discount rates. What does vary is the value of the multiplier $\bar{\beta}$.

In the model of subsection 5.1 the set of all feasible values for the various real quantities and the prices and incomes are given by the intersection of (2.1), (2.2), (2.3), (2.4) and (5.1). For the model of subsection 5.2, the feasible set is determined by replacing (5.1) with (5.9). We shall show that when it is assumed that all of the constraints are binding, the intersection of these constraints is identical in the two cases.

It is sufficient to consider the materials balance constraint (2.4) together with (5.1) and (5.9). Supposing (2.4) and (5.1) hold with equality, we obtain

$$\sum_i q_i (x_i - y_i) + \sum_i t_i y_i = b \qquad (5.15)$$

upon substitution of (2.4) into (5.1). Similarly we obtain

$$\sum_i p_i (x_i - y_i) + \sum_i t_i x_i = b \qquad (5.16)$$

when (5.9) is used. These are both equal to

$$\sum_i q_i x_i - \sum_i p_i y_i = 0. \qquad (5.17)$$

In other words, for fixed p and q, the set of all x, y and z which solve (2.4) and (5.17) with equality are identical to the x, y and z which solve either (2.4) and (5.1) with equality or (2.4) and (5.9) with equality. As a consequence, the optimal solutions do not depend upon the particular method of tax collection adopted[17].

[17] Some geometrical intuition for this result can be obtained by assuming there is only one good. By taking fixed values for p and q (and, hence, t), the boundaries of (5.1) and (5.2) yield planes in x-y-z space. The intersection of each of these planes with the plane given by assuming (2.4) holds with equality are identical.

What we are observing is the fact that when the overall constraint is the intersection of a series of constraints, it is possible to have the same overall constraint arise even if the constituent constraints are different. As the formulae for the decompositions of our discount rates are given by the gradients of the constituent constraints, this observation accounts for the different decompositions found in subsections 5.1 and 5.2.

Used with care, these alternative decompositions of δ_t might be of practical benefit in calculating bounds for δ_t. In discussing (5.8) and (5.14) we presented sufficient conditions to obtain upper and lower bounds for δ_t. As these bounds will differ for each of these two formulae, we can combine our results to obtain sharper bounds, in general, than if only (5.8) or (5.14) were used to bound δ_t.

In showing the equivalence of both (5.15) and (5.16) to (5.17) it is important that the public enterprise has access to the commodity tax revenue. If commodity tax revenues could only be used for financing the lump-sum incomes R^h and not for subsidizing the public enterprise, switching from taxes on producer transactions to taxes on consumer transactions will have real effects[18].

6. Conclusions

In these concluding remarks we wish to highlight what we believe are the most important contributions of this paper.

First, we have initiated a systematic study into the consequences for optimal discount rate formulae of altering the environment faced by a public firm which is subject to a financial constraint. We have seen that the variations in the financial constraints we have considered are captured by simple modifications in the basic discount rate formulae presented in section 3.

Second, we have explicitly used the discount rates implicitly defined by the social values of commodities. Because of the restrictions implied by optimality on the gradients of objectives and constraints, the use of these discount rates appears quite natural, and leads to formulae in the familiar weighted-sum form.

Third, we have seen in section 5 that when commodity taxes are levied it is possible to decompose the formulae for the discount rates of the public firm in alternative fashions which, in one case, would lead one to believe that they are related to the private pro-

[18] Hagen (1982) develops discount rate formulae when commodity taxes are present but the revenue is not available for the public enterprise's use.

ducer discount rates and, in the other case, would lead one to believe they are related to the consumer discount rates. However, when these tax revenues are used to subsidize the public enterprise, the optimal public discount rates are identical in both cases, suggesting that caution must be used in interpeting and applying these formulae.

Finally, we have considered the practical difficulties involved in calculating the appropriate discount rates. If there is more than one financial constraint, our formulae do not appear to be easily calculated nor are there natural bounds on the shadow discount rates. However, if there is only one financial constraint the situation improves, particularly if cross-elasticities are zero. In this case it is possible to calculate both upper and lower bounds for the shadow discount rates using only observable data, and these lower bounds are given by one of the private market rates. In these circumstances our formulae indicate what information is needed to calculate the bounds and these bounds limit the range of reasonable discount rates to apply to public production decisions[19]. Perhaps it is unrealistic to ask for stronger results than these.

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[19] This point was suggested by remarks made by Eytan Sheshinski at the ISPE conference.

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