The Core of a Transferable Utility Game as the Solution to a Public Good Market Demand Problem

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Abstract. The core of a monotonic transferable utility (TU) game is shown to be the set of prices that incentivize each individual to demand the grand coalition in a market demand problem in which the goods being demanded are coalitions viewed as excluable public goods. It is also shown that the core is the intersection of superdifferentials evaluated at the grand coalition of the covers of person-specific TU games derived from the original game. These characterizations of the core demonstrate how a market demand approach to coalition formation in the spirit of Baldwin and Klemperer [4, 5] is related to the approach to the core using the cover of a TU game and it superdifferential at the grand coalition developed by Shapley and Shubik [25], Aubin [3], and Danilov and Koshevoy [8].

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1. Introduction

In this article, we provide a novel interpretation of the core of a TU game using tools adapted from the theory of discrete convex optimization applied to the analysis of a market demand problem in which coalitions are thought of as being excludable public goods. In doing this, we build on an approach to the core by Shapley and Shubik [25], Aubin [3], and Danilov and Koshevoy [8] that employs the cover of a TU game and on work by Baldwin and Klemperer [4, 5] on the demand for indivisible private goods, which they have applied to a problem of partitioning individuals into coalitions.

For each possible coalition of the set of individuals $N = \{1, \ldots, n\}$, a transferable utility (TU) game v specifies the total utility—the coalition's value—that can be shared among its members [16, 19]. We interchangeably regard a coalition as being a subset S of N and as the vertex $\mathbf{1}_S$ of $\{0,1\}^n$ whose ith component is 1 if and only if $i \in S$, and use $\{0,1\}^n$ as the domain of v. A core allocation divides the total utility of the grand coalition N among the n individuals in such a way that no subgroup can receive a larger total utility on its own. A core allocation can be thought of as providing a way of fairly dividing the grand coalition's value. The core is defined by a system of linear equalities and inequalities. Necessary and sufficient conditions for the nonemptyness of the core were developed by Bondareva [7] and Shapley [22] using linear programming techniques.

A coalition can be thought of as being an excludable public good [21]. A public good is non-rivalrous in consumption; that is, the consumption of this good by one person does not preclude its consumption by the other individuals. In contrast, with a private good, a unit of this good consumed by one individual cannot be consumed by anybody else. A public good can be either excludable or non-excludable. If it is non-excludable, nobody can be prevented from consuming it. On the other hand, if it is excludable, it is possible to limit its consumption to any group of individuals. While all those who are not excluded consume a public good in common, they need not value it in the same way.

We show that the core can be characterized using market demands for coalitions viewed as being excludable public goods. Individual j is paid a price p_j for joining a coalition. The cost of coalition S is the sum of the prices paid to its members. If individual i is a member of coalition S, the benefit of S to him is the sum of the value of S and the price p_i paid to himself for belonging to it; this benefit is instead 0 if i does not belong to S. Faced with the prices $p = (p_1, \ldots, p_n)$, i demands those coalitions that maximize his net benefit—the benefit less the cost. We show that if p is nonnegative and its components sum to the value of the grand coalition, then p is in the core if and only each individual demands the grand coalition at these prices when the game is monotonic. Thus, the prices that incentivize every individual to demand the grand coalition are the core allocations, provided that such prices exist.

We also relate this characterization of the core to the superdifferential of the cover of v. The cover of v is the smallest linearly homogeneous, concave, real-valued function on the nonnegative orthant $\mathbb{R}^n_+$ that is larger than v when restricted to $\{0,1\}^n$. The cover of a TU game was first introduced by Shapley and Shubik [25], who proved that the core is nonempty if and only if the value of the cover at $\mathbf{1}_N$ is equal to $v(\mathbf{1}_N)$. Subsequently, Aubin [3] showed that when the core is nonempty, it is equal to the superdifferential of the cover at $\mathbf{1}_N$. We call this pair of results the Shapley–Shubik–Aubin Theorem.[1] We also show that the set of prices that induce individual i to demand the grand coalition is the superdifferential at $\mathbf{1}_N$ of the cover of a person-specific TU game. The core is the intersection of these superdifferentials.

Aubin [3] established the Shapley–Shubik–Aubin Theorem as a corollary to an analogous theorem for fuzzy games [2, 3] in which individuals have degrees of participation in a coalition, but he only sketched the proof. We provide a complete direct proof of this theorem.

In a model with multiple indivisible private goods, Baldwin and Klemperer [4, 5] used tropical geometry [14] to investigate the existence of prices for which aggregate demand is equal to the supply of an arbitrary bundle of the goods. They showed that their market demand analysis can be applied to the problem of partitioning a set of individuals into coalitions that exhibit the stability properties of the core. They did not regard coalitions as being public goods; instead, they treated the individuals as being indivisible private goods which are demanded by special agents who form the coalitions.

In Section 2, we introduce TU games and the core. In Section 3, we show that the core consists of the prices that induce each individual to choose the grand coalition in a coalition market demand problem. In Section 4, we formally state and prove the Shapley–Shubik–Aubin Theorem about the existence and characterization of the core of a TU game. This is followed in Section 5 by our demonstration that the market demands can be used to characterize the core as the intersection of the superdifferentials at the grand coalition of person-specific TU games.

[1] A similar approach to characterizing the core has been employed by Danilov and Koshevoy [8]. Related constructions have been used to study the core of a game without side payments. See, for example, [3] and [6].

We discuss related literature in Section 6. Finally, in Section 7, we offer some concluding remarks.

2. The Core of a TU game

The set of *individuals* is $N = \{1, \dots, n\}$, where $n \geq 2$. A *coalition* is a subset of N. The coalition N is the *grand coalition*. As noted above, for the coalition $S \subseteq N$, we can identify S with the vector $\mathbf{1}_S \in \mathbb{R}^n$ whose ith coordinate is 1 if $i \in S$ and is 0 otherwise. With this representation of a coalition, the set of possible coalitions are the vertices of the Boolean hypercube $\{0,1\}^n$.

A *characteristic function* is a function $v\colon \{0,1\}^n \to \mathbb{R}$ for which $v(\mathbf{1}_\varnothing) = 0$. The value $v(\mathbf{1}_S)$ is the total utility that may be shared among the members of S. In other words, utility is transferable. We refer to $v(\mathbf{1}_S)$ as the coalition's *value*. For a fixed set of individuals N, a *transferable utility* (TU) game is defined by its characteristic function v.

An *allocation* is a vector $x \in \mathbb{R}^n$ specifying the utility of each of the n individuals. The *core* $C(v)$ of the TU game v is the set of all allocations for which

$$x \cdot \mathbf{1}_N = v(\mathbf{1}_N) \tag{1}$$

and

$$x \cdot \mathbf{1}_S \geq v(\mathbf{1}_S),\ \forall S \subseteq N. \tag{2}$$

Informally, a core allocation divides the value of the grand coalition among all of the individuals in such a way that any subgroup of individuals receives no less in total than what it could achieve by itself. A core allocation is undominated in the sense that no coalition acting on its own can provide larger utilities to all of its members.

The core is a compact convex polyhedron whose dimension is at most $n-1$. Bondareva [7] and Shapley [22] have identified a necessary and sufficient condition for a TU game to have a nonempty core in terms of balanced families of coalitions. A family $\mathscr{S}$ of coalitions is *balanced* if there exists for each $S \in \mathscr{S}$ a *balancing coefficient* $\lambda_S \in (0,1]$ for which $\sum_{S \in \mathscr{S}} \lambda_S \mathbf{1}_S = \mathbf{1}_N$. Equivalently, a family $\mathscr{S}$ is balanced if the vector $\mathbf{1}_N$ is in the positive cone generated by $\{\mathbf{1}_S \mid S \in \mathscr{S}\}$. A TU game v is *balanced* if for every balanced family $\mathscr{S}$ and associated balancing coefficients $\{\lambda_S \mid S \in \mathscr{S}\}$,

$$\sum_{S \in \mathscr{S}} \lambda_S v(\mathbf{1}_S) \leq v(\mathbf{1}_N). \tag{3}$$

Theorem 1 is the Bondareva–Shapley Theorem.

Theorem 1. *A TU game v has a nonempty core if and only if it is balanced.*

We restrict attention to monotonic TU games. *Monotonicity* is the requirement that

$$v(\mathbf{1}_S) \leq v(\mathbf{1}_T),\ \forall S,T \subseteq N \text{ such that } S \subseteq T.$$

Because $v(\mathbf{1}_\varnothing) = 0$, monotonicity implies that values are nonnegative,

$$v(\mathbf{1}_S) \geq 0,\ \forall S \subseteq N.^2$$

For further discussion of cores of TU games, see [16, 19].

3. Market Demands for Coalitions

We now formulate the problem of selecting a coalition as a problem of market demand for an excludable public good. While a coalition is only of value to someone if he is a member of it, the coalition itself is a public good.

Each individual is paid a price for joining a coalition. Prices are denominated in units of utility. Let $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ be the vector of these prices, where p_j is the price paid to individual j for being in a coalition. At this stage, we do not preclude some of these prices from being negative. However, we shall subsequently require them to be nonnegative. Thus, for any coalition $S \subseteq N$, the *outlay* (the "cost") required to form this coalition at the prices p is

$$O(S,p) = p \cdot \mathbf{1}_S. \tag{4}$$

For any $i \in N$ and any $S \subseteq N$, the *benefit* (the "utility") of S for i is

$$U^i(S,p) = \begin{cases} v(\mathbf{1}_S) + p_i, & \text{if } i \in S; \\ 0, & \text{if } i \notin S. \end{cases} \tag{5}$$

If $i \in S$, his benefit is the sum of the value $v(\mathbf{1}_S)$ of S and the amount being paid to himself for being part of this coalition. He gets no benefit from any coalition to which he does not belong.

For given prices, i chooses the coalitions that maximize his net benefit:

$$\max_{S \subseteq N} \left[U^i(S,p) - O(S,p) \right]. \tag{6}$$

[2] Only Lemma 3 and, hence, Theorem 4 use monotonicity in an essential way. For our other results, nonnegativity of v is sufficient.

His *coalition demand* at these prices is

$$D^i(p) = \arg\max_{S \subseteq N} \left[U^i(S,p) - O(S,p) \right]. \tag{7}$$

The optimization problem (6) is not meant to describe the actual way in which a coalition is formed. Rather, it is a hypothetical construct designed to provide an alternative interpretation of the core of a TU game. An alternative interpretation of the coalition market demand problem (6) in which i does not make payments to himself is provided in Section 5.

We are interested in determining whether there exist nonnegative prices p^* such that $p^* \cdot \mathbf{1}_N = v(\mathbf{1}_N)$ and $D^i(p^*) = N$ for all $i \in N$. Provided that v is monotonic, Theorem 2 shows that the set of all such prices is the core of v.

Theorem 2. *Let v be a monotonic TU game. If p^* is nonnegative with $p^* \cdot \mathbf{1}_N = v(\mathbf{1}_N)$, then $p^* \in C(v)$ if and only if $N \in D^i(p^*)$ for all $i \in N$.*

Proof. (a) Suppose that $p^* \in C(v)$. Because, by assumption, $p^* \cdot \mathbf{1}_N = v(\mathbf{1}_N)$, equation (2) with $x = p^*$ is equivalent to

$$0 = v(\mathbf{1}_N) - p^* \cdot \mathbf{1}_N \geq v(\mathbf{1}_S) - p^* \cdot \mathbf{1}_S, \ \forall S \subseteq N. \tag{8}$$

By (4) and (5) we know that

$$v(\mathbf{1}_N) - p^* \cdot \mathbf{1}_N = U^i(N,p^*) - O(N,p^*) - p_i^*$$

and hence for any $i \in N$, the inequality in (8) is equivalent to

$$U^i(N,p^*) - O(N,p^*) \geq v(\mathbf{1}_S) + p_i^* - p^* \cdot \mathbf{1}_S, \ \forall S \subseteq N. \tag{9}$$

If $i \in S$, the right-hand side of (9) is $U^i(S,p^*) - O(S,p^*)$. Because v is monotonic, $v(\mathbf{1}_S) \geq 0$. Thus, if $i \notin S$, because both $v(\mathbf{1}_S)$ and p_i^* are nonnegative, the right-hand side of (9) is no less than $U^i(S,p^*) - O(S,p^*)$. Hence, $N \in D^i(p^*)$.

(b) Suppose that $N \in D^i(p^*)$ for all $i \in N$. Because (2) trivially holds when $S = \varnothing$, we only need to show that it also holds for the nonempty coalitions. By (7), we have

$$U^i(N,p^*) - O(N,p^*) \geq U^i(S,p^*) - O(S,p^*)$$

for all $i \in N$ and all $S \subseteq N$. Consider any $S \neq \varnothing$ and any $i \in S$. Then, $U^i(S,p^*) = v(\mathbf{1}_S) + p_i^*$ and so

$$v(\mathbf{1}_N) + p_i^* - p^* \cdot \mathbf{1}_N \geq v(\mathbf{1}_S) + p_i^* - p^* \cdot \mathbf{1}_S,$$

which implies that

$$v(\mathbf{1}_N) - p^* \cdot \mathbf{1}_N \geq v(\mathbf{1}_S) - p^* \cdot \mathbf{1}_S.$$

Given our assumption that $v(\mathbf{1}_N) = p^* \cdot \mathbf{1}_N$, we have

$$0 \geq v(\mathbf{1}_S) - p^* \cdot \mathbf{1}_S,$$

which confirms that (2) holds for this S. Because $S \neq \varnothing$ was chosen arbitrarily and (1) holds by hypothesis, we conclude that p^* is in the core of v. □

We have thus shown that the core, when it exists, is the set of prices that induce each individual to choose the grand coalition in the coalition market demand problem (6) when prices are normalized to sum to the value of the grand coalition.

4. The Shapley–Shubik–Aubin Theorem

In this section, we formally state and prove the Shapley–Shubik–Aubin Theorem.

We begin by generalizing the concept of balanced families discussed in Section 2 in the following way. For $x \in \mathbb{R}^n_+$, a *balancing set for* x is a collection $L_x = \{\lambda_S \geq 0\}_{\varnothing \neq S \subseteq N}$ of nonnegative coefficients indexed by the coalitions such that

$$x = \sum_{\substack{S \subseteq N \\ S \neq \varnothing}} \lambda_S \mathbf{1}_S.$$

Note that for $x = \mathbf{1}_\varnothing = (0, 0, \ldots, 0)$, the balancing coefficients must all be 0. Let $\mathscr{L}_x$ be the set of all balancing sets for x. For the game $v\colon \{0,1\}^n \to \mathbb{R}$, the *cover* of v [25] is the function $f_v\colon \mathbb{R}^n_+ \to \mathbb{R}$ defined by setting

$$f_v(x) = \sup_{L_x \in \mathscr{L}_x} \sum_{S \subseteq N} \lambda_S v(\mathbf{1}_S), \ \forall x \in \mathbb{R}^n_+. \tag{10}$$

It is easy to see that the collection of balancing sets $\mathscr{L}_x$ is a compact subset of $\mathbb{R}^{2^n - 1}$, so the sup exists and is achieved. Note that this construction implies that

$$v(\mathbf{1}_S) \leq f_v(\mathbf{1}_S), \ \forall S \subseteq N.$$

Informally, the function f_v defined in (10) is the smallest linearly homogeneous, concave, real-valued function on $\mathbb{R}^n_+$ that is larger than v on $\{0,1\}^n$.

A function $f\colon \mathbb{R}^n_+ \to \mathbb{R}$ is *concave* if

$$\alpha f(p) + (1-\alpha) f(q) \leq f(\alpha p + (1-\alpha) q), \ \forall \alpha \in [0,1], \ \forall p, q \in \mathbb{R}^n_+,$$

and it is *linearly homogeneous* if

$$f(kx) = kf(x),\ \forall k \in \mathbb{R}_+,\ \forall x \in \mathbb{R}^n_+.$$

We now demonstrate that f_v satisfies these two properties.

Lemma 1. *The cover f_v of v is concave.*

Proof. Suppose that the balancing set for p that achieves the value $f_v(p)$ is $\{\lambda_S\}$ and the balancing set for q that achieves $f_v(q)$ is $\{\gamma_S\}$. Then, we know that $p = \sum_{S\subseteq N} \lambda_S \mathbf{1}_S$ and $q = \sum_{S\subseteq N} \gamma_S \mathbf{1}_S$. Thus,

$$\begin{aligned}\alpha p + (1-\alpha) q &= \sum_{S\subseteq N} \alpha \lambda_S \mathbf{1}_S + \sum_{S\subseteq N} (1-\alpha)\gamma_S \mathbf{1}_S \\ &= \sum_{S\subseteq N} (\alpha \lambda_S + (1-\alpha)\gamma_S)\mathbf{1}_S,\end{aligned}$$

which shows that $\{\alpha \lambda_S + (1-\alpha)\gamma_S\}$ is a balancing set for $\alpha p + (1-\alpha)q$. By the definition of f_v, we know that

$$\begin{aligned}f_v(\alpha p + (1-\alpha)q) &\geq \sum_{S\subseteq N} (\alpha \lambda_S + (1-\alpha)\gamma_S) v(\mathbf{1}_S) \\ &= \sum_{S\subseteq N} \alpha \lambda_S v(\mathbf{1}_S) + \sum_{S\subseteq N} (1-\alpha)\gamma_S v(\mathbf{1}_S) \\ &\geq \alpha f_v(p) + (1-\alpha) f_v(q).\end{aligned}$$

□

Lemma 2. *The cover f_v of v is linearly homogeneous.*

Proof. The result for the case in which $k = 0$ follows that fact that $f_v(\mathbf{1}_\varnothing) = 0$. For $k > 0$, it is obvious that $L_x = \{\lambda_S\}$ is a balancing set for x if and only if $L_{kx} = \{k\lambda_S\}$ is a balancing set for kx. It then follows that

$$f_v(kx) = \sup_{L_{kx}\in\mathscr{L}_{kx}} \{k\lambda_S\} = k \sup_{L_x\in\mathscr{L}_x} \{\lambda_S\} = k f_v(x).$$

□

Because f_v is concave and linearly homogeneous, it is also continuous. Associated with any concave function are its supergradients and superdifferentials [20].

A vector $x^* \in \mathbb{R}^n$ is a *supergradient* of a continuous concave function $f\colon \mathbb{R}^n_+ \to \mathbb{R}$ at x if

$$f(z) \leq f(x) + x^* \cdot (z - x),\ \forall z \in \mathbb{R}^n_+.$$

A supergradient exists at every point in the relative interior of the domain of a concave function [20, Theorem 23.4]. The *superdifferential* of f at x, denoted $\partial f(x)$, is the set of supergradients of f at x. This superdifferential is a compact convex set. If f is differentiable at x, then the unique supergradient is simply the gradient.

Theorem 3 is the Shapley–Shubik–Aubin Theorem.

Theorem 3. *Let v be a monotonic TU game and f_v be the cover of v. (a) $C(v) \neq \varnothing$ if and only if $v(\mathbf{1}_n) = f_v(\mathbf{1}_N)$. (b) If $C(v) \neq \varnothing$, then $C(v) = \partial f_v(\mathbf{1}_N)$.*

Proof. The proof strategy is as follows. First, we show that $C(v) = \varnothing$ if $v(\mathbf{1}_N) \neq f_v(\mathbf{1}_N)$. Second, we show that if $v(\mathbf{1}_N) = f_v(\mathbf{1}_N)$, then $\partial f_v(\mathbf{1}_N) \subseteq C(v)$. Third, we show that if $v(\mathbf{1}_N) = f_v(\mathbf{1}_N)$, then $C(v) \subseteq \partial f_v(\mathbf{1}_N)$. Because the superdifferential of a concave function is nonempty at each point in its domain, the second and third steps in the proof imply that $C(v) \neq \varnothing$ if $v(\mathbf{1}_N) = f_v(\mathbf{1}_N)$.

(i) Suppose that $v(\mathbf{1}_N) \neq f_v(\mathbf{1}_N)$. Then, for some balancing set $\{\lambda_S\}$ for $\mathbf{1}_N$, we must have $v(\mathbf{1}_N) < \sum_{S \subseteq N} \lambda_S v(\mathbf{1}_S)$. The collection $\{S \,|\, \lambda_S > 0\}$ thus forms a balanced set that violates (3) and, hence, by Theorem 1, we know that v has an empty core.

(ii) Now suppose that $v(\mathbf{1}_N) = f_v(\mathbf{1}_N)$. Let $p \in \partial f_v(\mathbf{1}_N)$. We show that p is in the core of v.

From the definition of a supergradient, we have for all $z \in \mathbb{R}^n_+$ that

$$f_v(z) \leq f_v(\mathbf{1}_N) + p \cdot (z - \mathbf{1}_N),$$

which implies that

$$f_v(z) - p \cdot z \leq f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N. \tag{11}$$

We now show that the right-hand side of (11) is equal to 0. By choosing $z = \mathbf{1}_\varnothing = (0, 0, \dots, 0)$, (11) implies that

$$0 \leq f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N. \tag{12}$$

Now consider $z = 2\mathbf{1}_N$. Because f_v is linearly homogeneous, we have that

$$f_v(2\mathbf{1}_N) = 2 f_v(\mathbf{1}_N) \tag{13}$$

and, therefore,

$$f_v(2\mathbf{1}_N) - p \cdot 2\mathbf{1}_N = 2[f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N]. \tag{14}$$

Equations (11)–(14) jointly imply that

$$0 \leq 2[f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N] \leq f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N,$$

which proves that $f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N = 0$. Hence, (1) holds for the vector p because, by assumption, $f_v(\mathbf{1}_N) = v(\mathbf{1}_N)$.

Because $f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N = 0$, (11) reduces to

$$f_v(z) - p \cdot z \leq 0$$

for all $z \in \mathbb{R}^n_+$. Choose $z = \mathbf{1}_S$ for some coalition S. From the construction of f_v, we know that $f_v(\mathbf{1}_S) \geq v(\mathbf{1}_S) \geq 0$ and, hence, that

$$v(\mathbf{1}_S) - p \cdot \mathbf{1}_S \leq f_v(\mathbf{1}_S) - p \cdot \mathbf{1}_S \leq 0.$$

Therefore,

$$v(\mathbf{1}_S) \leq p \cdot \mathbf{1}_S,$$

which establishes (2) for the vector p confirming that $p \in C(v)$.

(iii) Finally, suppose that $v(\mathbf{1}_N) = f_v(\mathbf{1}_N)$and $p \in C(v)$. We need to show that $p \in \partial f_v(\mathbf{1}_N)$. Consider any $z \in \mathbb{R}^n_+$. By the definition of f_v in (10), there exists a balancing set $\{\lambda_S\}$ for z such that

$$f_v(z) = \sum_{S \subseteq N} \lambda_S v(\mathbf{1}_S).$$

We then have that

$$p \cdot z = p \cdot \sum_{S \subseteq N} \lambda_S \mathbf{1}_S = \sum_{S \subseteq N} \lambda_S (p \cdot \mathbf{1}_S) \geq \sum \lambda_s v(\mathbf{1}_S) = f_v(z), \tag{15}$$

where the inequality follows because p is in the core and so satisfies (2). Because p is in the core, it also satisfies (1). Hence, $p \cdot \mathbf{1}_N = v(N) = f_v(\mathbf{1}_N)$. Using (15), it then follows that

$$f_v(\mathbf{1}_N) - p \cdot \mathbf{1}_N + p \cdot z \geq f_v(z)$$

or, equivalently,

$$f_v(\mathbf{1}_N) + p \cdot (z - \mathbf{1}_N) \geq f_v(z),$$

which shows that $p \in \partial f_v(\mathbf{1}_N)$.

□

5. Market Demands and the Core

In Section 3, we showed how to interpret vectors in the core of a TU game v as prices paid to individuals for their participation in a coalition that incentivize each individual to choose the grand coalition in a market demand game in which the coalitions are treated as excludable public goods. In Section 4, we established the Shapley–Shubik–Aubin Theorem, which identifies when the core is nonempty and characterizes it in terms of the superdifferential of the cover of v. In this section, we relate these two characterizations of the core.

Given a monotonic TU game v and an individual $i \in N$, we define a new monotonic TU game v^i by, for each $S \subseteq N$, setting

$$v^i(\mathbf{1}_S) = \begin{cases} v(\mathbf{1}_S), & \text{if } i \in S; \\ 0, & \text{if } i \notin S. \end{cases} \tag{16}$$

The game v^i gives individual i the value of any coalition to which he belongs and nothing from the formation of a coalition that does not include him.

Note that the coalition market demand problem (6) can be equivalently written as

$$\max_{S \subseteq N} \left[v^i(\mathbf{1}_S) - \sum_{j \in S \setminus \{i\}} p_j \right]. \tag{17}$$

Thus, for any coalition S in which i is a member, i can be thought of as keeping for himself the value of S net of the payments made to the other members of S for their participation. If i does not belong to S, he pays its members to form S, but receives no benefit himself.

Lemma 3. *Let v be a monotonic TU game. The game v^i defined in* (16) *has a nonempty core for every individual $i \in N$.*

Proof. Let $p^i = (p^i_1, \ldots, p^i_n)$ be the vector

$$p^i_j = \begin{cases} v(\mathbf{1}_N), & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases}$$

It follows from the the definition of v^i and the monotonicity of v (and, hence, of v^i) that p^i is in the core of v^i. □

For each $i \in N$, let

$$\psi^i(\mathbf{1}_N) = \{p \in \mathbb{R}^n \,|\, p \geq 0,\ N \in D^i(p),\ \text{and } p \cdot \mathbf{1}_N = v(\mathbf{1}_N)\}.$$

The set $\psi^i(\mathbf{1}_N)$ consists of the prices that incentivize individual i to demand the grand coalition in the coalition market demand problem (6) or, equivalently, (17). We associate with v^i its cover $f_{v^i}\colon \mathbb{R}^n_+ \to \mathbb{R}$ as in (10). As we now show, the core of v^i is both the set $\psi^i(\mathbf{1}_N)$ and the superdifferential of f_{v^i} at $\mathbf{1}_N$.

Theorem 4. *Let v be a monotonic TU game. For any $i \in N$, let v^i be the TU game defined in* (16) *and f_{v^i} be its cover. Then,*

$$C(v^i) = \psi^i(\mathbf{1}_N) = \partial f_{v^i}(\mathbf{1}_N).$$

Proof. Given how v^i is defined in (16), for all $S \subseteq N$, $U^i(S,p) - O(S,p)$ in (5) takes on the same value whether it is defined using v or v^i. Thus, the proof that $C(v^i) \subseteq \psi^i(\mathbf{1}_N)$ is exactly the same as part (a) of the proof of Theorem 2.

Now, suppose that $p \in \psi^i(\mathbf{1}_N)$. By hypothesis, p satisfies (1). It remains to show that it also satisfies (2). There are two cases. First, if $i \notin S$, then $p \cdot \mathbf{1}_S \geq 0 = v^i(S)$, where the inequality follows from the assumption that $p \geq 0$. If $i \in S$, part (b) of the proof of Theorem 2 applies and, hence, (2) holds.

We have shown that $C(v^i) = \psi^i(\mathbf{1}_N)$. By Lemma 3, $C(v^i) \neq \varnothing$. Hence, by Theorem 3, $C(v^i) = \psi^i(\mathbf{1}_N)$ is also equal to $\partial f_{v^i}(\mathbf{1}_N)$. □

For a monotonic TU game v, we know from Theorem 2 that the core is the set of prices that induce each individual to demand the grand coalition in their coalition market demand problems. This set is $\cap_{i \in N} \psi^i(\mathbf{1}_N)$. Thus, it follows immediately from Theorem 4 that the core is the intersection of the superdifferentials of the covers f_{v^i} associated with the person-specific TU games v^i.

Theorem 5. *Let v be a monotonic TU game. For every $i \in N$, let v^i be the TU game v^i defined in* (16) *and f_{v^i} be its cover. Then,*

$$C(v) = \bigcap_{i=1}^{n} \partial f_{v^i}(\mathbf{1}_N). \tag{18}$$

Theorem 5 does not assume that the core is nonempty. If it is empty, then there are no prices that induce everybody to demand the grand coalition, and so the intersection of the superdifferentials in (18) is empty.

6. Remarks on the Literature

Shapley and Shubik [25] introduced the concept of the cover of a TU game v and used it to establish part (a) of Theorem 3. They did not offer a characterization of the core of v (when it exists) in terms of the superdifferential of the cover of v at the grand coalition. Our proof of part (a) of Theorem 3 makes use of this characterization.

Aubin [2, 3] was primarily concerned with with the existence and characterization of the core for cooperative fuzzy games. In the transferable utility case, a fuzzy game is defined by a linearly homogenous function v that assigns a value to each vector $\tau \in [0,1]^n$. The ith component of τ is the degree to which individual i participates. Aubin used the linearly homogenous extension of v to $\mathbb{R}^n_+$ to determine when the core of a fuzzy TU game is nonempty and to characterize a nonempty core in terms of the superdifferential of this function at $\mathbf{1}_N$. For a standard TU game v, he showed that its cover restricted to $[0,1]^n$ can be thought of as being a fuzzy TU game, from which his results about the core of v follow.

Shapley and Shubik [25, p. 16] have noted that the balancing weight λ_S can be interpreted as the degree of participation of the members of S in this coalition, thereby connecting balancedness with fuzzy games. With this interpretation, the maximization problem (10) defining the cover of v identifies how each individual's participation in each coalition of which he is a member should be optimally assigned.

Danilov and Koshevoy [8] analyzed TU games using an approach that is closely related to that of Aubin [3]. Using Choquet integrals, they extended a TU game v to a linearly homogenous function $\tilde{v}$ on all of $\mathbb{R}^n$ that agrees with v at each of the vectors $\mathbf{1}_S$. They showed that if $\partial\tilde{v}(\mathbf{1}_N)$ exists, then it is equal to the core of v. For a standard TU game, in Aubin's approach, the analogue to $\tilde{v}$ is the cover f_v, which is linearly homogenous and concave by construction. In contrast, $\tilde{v}$ need not be concave, and so a supergradient at $\mathbf{1}_N$ need not exist. Danilov and Koshevoy focused on games that are supermodular.[3] Shapley [24] called such games *convex*. Convex games always have a nonempty core. For convex games, Danilov and Koshevoy proved that $\tilde{v}$ is concave and the superdifferential of $\tilde{v}$ at $\mathbf{1}_N$ is equal to the core. They also provided a geometric description of the core in terms of Minkowski sums and differences of the cores of certain simple games.

Baldwin and Klemperer [4, 5] modeled coalition formation as the solution to a market demand problem. Coalitions are allowed to form only if they result

[3] See [12] for a definition and some other properties that are equivalent to supermodularity.

in a pre-specified partition $\mathscr{N}$ of the set of individuals N. As is the case here, there is a price vector p whose jth component is the price paid to individual j for joining a coalition. Associated with each nonempty coalition S in $\mathscr{N}$, there is a coalition agent who must choose between S and the empty set. If this agent chooses S, he keeps its value (which in the Baldwin–Klemperer model is the sum of the individual values of the members of S for that coalition) less the sum of the prices paid to the members of S for joining S and transferring their values to the coalition agent. Each person is thought of as a private person-good that can only be "consumed" in the quantity 0 or 1 by one coalition agent. Given $\mathscr{N}$, the aggregate demand on the part of all of the coalition agents for any individual j is simply the demand by the coalition agent for the element in $\mathscr{N}$ that j belongs to. Baldwin and Klemperer investigated whether there exist prices p such that the coalition agents demand the partition $\mathscr{N}$. In our model, there are no coalition agents and the demand is for an excludable public good (the coalition), not for private goods (the individuals viewed separately).

The model of coalition market demand considered by Baldwin and Klemperer [5] is a special case of their model of the demand for indivisible private goods. A bundle of k private goods is a k-tuple in $\mathbb{Z}^k$. Their unimodularity theorem provides a necessary and sufficient condition for any bundle of goods to be the aggregate consumer demand at some prices.[4] A partition of N in the coalition formation problem corresponds to a bundle of goods in the market demand problem.

In a *competitive equilibrium* for divisible private goods, prices equate the demand and supply (aggregated over all consumers and firms) of each good [1]. The equilibria considered by Baldwin and Klemperer [5] are competitive equilibria for an economy with indivisible private goods in which each of the goods is in fixed supply. Danilov, Koshevoy, and Murota [9] and Murota [17] have investigated when a competitive equilibrium exists for indivisible private goods when there is also a divisible private good (money) and supply is price sensitive. For simplicity, we only describe the fixed-supply case.

The sufficient conditions in [9, 17] for the existence of an equilibrium with indivisible private goods are developed using a convexified economy in which constructions related to the cover of a TU game are employed. In particular, the real-valued utility function U_i of individual i defined on $\mathbb{Z}^k$ is concavified by a function $\tilde{U}_i$ on $\mathbb{R}^k$ that satisfies the properties of a cover except for linear homogeneity. Murota's [17] sufficient condition for an equilibrium requires that the

[4]See [4, 5] for a definition of unimodularity. An alternative proof of Baldwin and Klemperer's unimodularity theorem using linear programming has been developed by Tran and Yu [26].

concavified utility function agrees with the original utility function on $\mathbb{Z}^k$, whereas in the Aubin–Shapley–Shubik Theorem, agreement is only required at $\mathbf{1}_N$. An equilibrium price vector is a common supergradient of (i) all of the individual utility functions at their individual bundles and (ii) the sum of the utility functions at the aggregate bundle [17, Sec. 11.4]. In contrast, for us, the supergradients are all evaluated at the same point, $\mathbf{1}_N$. This contrast is of fundamental importance. In the indivisible private goods case, the Minkowski sum of the individual demands at given prices need not be a discrete convex set [17, Sec. 11.2], which is why conditions like those used in [5, 9, 17] are needed to ensure that an equilibrium exists. In our problem, there is no analogue of an aggregate demand, so this issue does not arise.

Lindahl [13] proposed a market-like mechanism for allocating divisible private and public goods in which the price for a private good is common to everyone, whereas the prices for a non-excludable public good are personalized and sum to the production price. In a *Lindahl equilibrium*, prices are such that the demand and supply of each good, both private and public, are equated. In equilibrium, each individual can demand different quantities of a private good, but they must all demand the same quantity of a public good. Fabre-Sender [10] and Foley [11] have shown that the existence of a Lindahl equilibrium is equivalent to the existence of a competitive equilibrium in a model with only private goods by treating the consumption of a public good by an individual as a private good that only this person demands (see [15]). Private person-goods play a somewhat similar role in Baldwin and Klemperer's model of coalition formation in that such goods are only demanded by one coalition agent. In our framework, a public good is a coalition, which is an indivisible good. For the coalition S, its price is the outlay $O(S,p)$. In contrast to Lindahl pricing, this price is not personalized.

7. Conclusion

In this article, we have shown that the core of a monotonic TU game can be interpreted as being the set of prices that induce each individual to choose the grand coalition in a market demand problem in which the coalitions are interpreted as being excludable public goods. We have also shown that the core is equal to the intersection of the superdifferentials at $\mathbf{1}_N$ of the covers of person-specific TU games derived from the original game. We have therefore demonstrated how a market demand approach to coalition formation in the spirit of Baldwin and Klemperer [4, 5] is related to the approach to the core using the cover of a TU game and its superdifferential at the grand coalition developed by Shapley and

Shubik [25], Aubin [3], and Danilov and Koshevoy [8].

Our analysis also sheds some light on the problem of partitioning individuals into coalitions and on the core of a non-transferable utility (NTU) game. In the case of coalition partitions, our approach can be applied to the problem of incentivizing individuals to sort themselves into the coalitions in some pre-specified partition $\mathscr{N}$. As noted in Section 6, Baldwin and Klemperer [4, 5] used a market demand approach to this problem in which coalition agents are the demanders and the goods are private person-goods. We can instead view each coalition S in the partition as an excludable public good and, as we have done here, regard the price p_j as being the amount that individual j must be paid to join a coalition. With this interpretation, the problem is to find a price vector p with the property that the individuals assigned to S by the partition $\mathscr{N}$ demand S at these prices. This can be done by reinterpreting S as being the grand coalition for the members of S and replacing v by its restriction to S. By Theorem 5, the core of this subgame is given by (18), but with the intersection only applying to members of S.

With a TU game, $v(\mathbf{1}_S)$ is a scalar—the aggregate utility that can be divided among the individuals in the coalition S should it form. With an NTU game, $v(\mathbf{1}_S)$ is instead the *vectors* of individual utilities that are achievable with S. As is the case here, for an NTU game, we can regard a coalition as an excludable public good. However, we cannot use (5) to define the utility of S for individual i because $v(\mathbf{1}_S)$ is not a scalar. Nevertheless, our analysis can be applied to an associated TU game in which the utility vectors in $v(\mathbf{1}_S)$ are weighted and summed. Specifically, as proposed by Shapley [23], for the NTU game v, we can associate with it a TU game using a weight vector $w = (w_1, \dots w_n)$. This game has a characteristic function v^w for which $v^w(\mathbf{1}_S)$ is the maximum value of $w \cdot u$ for $u \in v(\mathbf{1}_S)$. In effect, the set $v(\mathbf{1}_S)$ is expanded so that its upper boundary is given by the utility vectors whose w-weighted sum is equal to $v^w(\mathbf{1}_S)$. That is, it is supposed that "fictitious transfers" can take place on a one-to-one basis once utilities have been rescaled using the weights w. The value $v^w(\mathbf{1}_S) + p_i$ can be used in (5) to define i's utility for any coalition S that he is a member of. An allocation in the core of v^w need not be in the core (expressed in the non-weighted utilities) of the original NTU game. Various conditions have been developed that ensure that it is for some choice of the weights. However, in general, this procedure only identifies a subset of an NTU game's core when it is nonempty.[5] It is an open question as to whether weighted-utility TU games can be used to characterize the core of an NTU game,

[5] Aubin [3] used the superdifferential at $v(\mathbf{1}_N)$ of a fuzzy version of v^w for an appropriate choice of weights w to identify these core allocations.

not just a subset of it.[6]

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